Last time:

5d $\mathcal{N}=1$ $SU(2)$ gauge theory
with $N_f < 8$ hyper-multiplets

$\rightarrow$ effective gauge coupling:

$$\frac{1}{g_{eff}^2} = \frac{1}{g^2} + 8\phi - \frac{1}{2}\sum_{i=1}^{N_f} |\phi - m_i| - \frac{1}{2}\sum_{i=1}^{N_f} |\phi + m_i|$$

conformal fixed point at:

$$g = \infty, \quad \phi, m_i = 0$$

This story generalizes to $Sp(N)$ gauge theory with N_a anti-sym matter fields and N_f hyper-multiplets in the fundamental

$\rightarrow$ effective gauge coupling:

$$(g_{eff}^{-2})_{ii} = 2\left[(N-i)a_i + \sum_{k=1}^{i-1} a_k\right](1-N_a) + a_i(8-N_f)$$

$$(g_{eff}^{-2})_{i<j} = 2(1-N_a)a_j$$

$\rightarrow$ only positive semi-definite for $N_a=1$, $N_f \leq 7$

$\rightarrow$ SCFT fixed point in these cases

$Sp(1) = SU(2)$ is just a special case as anti-sym hyper is gauge singlet and decouples

Superconformal index

supercharges: Q_m^A, S^m_A

m is $SO(5)$ rotation index

A is $SU(2)$ R-symmetry index

We have the commutation relation:

$$\{Q_m^A, S^n_B\} = \delta^n_m \delta^A_B D + 2\delta^A_B M_m{}^n - 3\delta^n_m R_B{}^A$$

Choose supercharge

$$Q \equiv Q^{A=1}_{m=2}$$

→ Compute superconformal index, which counts BPS states which are annihilated by both Q and $S = Q^\dagger$

→ count $\frac{1}{8}$ BPS states

$$\Delta = \{Q, S\} = \Delta_0 - 2j_1 - 3R$$

Δ_0: energy / dilatation

j_1, j_2: Cartan generators of $SU(2)_1 \times SU(2)_2 \subset Sp(2) = SO(5)$

R : Cartan generator of $SU(2)_R$

Each of these Cartan generators of $F(4)$ commutes with supercharges Q and S

→ chemical potentials: $e^{-\beta}$, $x = e^{-\gamma_1}$, $y = e^{-\gamma_2}$

instanton number k also commutes with the supercharges $\rightarrow$ chemical pot. : q

Cartan generators of flavor symmetry group $SO(2N_f)$: H_i $(i = 1, 2, \ldots, N_f)$ have chemical potential e^{-m_i}

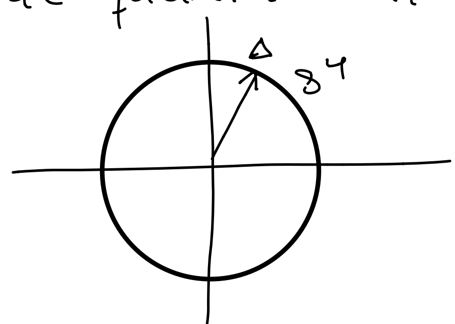
$\rightarrow$ Superconformal index:

$$I(x, y, m_i, q)$$

$$= \mathrm{tr}\left[(-1)^F e^{-\beta\{Q, S\}} x^{2(j_1+R)} y^{2j_2} e^{-i\sum_i H_i m_i} q^k\right]$$

where F is the fermion number operator

Trace is taken over the Hilbert space on S^4 after radial quantization.



$\rightarrow$ only states with $\Delta = 0$ contribute (other states with $\Delta > 0$ cancel out due to $(-1)^F$)

$\rightarrow$ index does not depend on chemical potential $e^{-\beta}$.

Can be computed through "Localization"

- Index for $Sp(1)$ gauge group:

consists of perturbative part and instanton part

1) perturbative part:

obtain spectrum for the global $SO(2N_f)$ symmetry, i.e. $U(1)_I$ charge $k=0$

2) Instanton part:

spectrum is enhanced to E_{N_f+1}

$U(1)_I$ part provides an extra Cartan and thus leads to symmetry enhancement

$$SO(2N_f) \times U(1)_I \rightarrow E_{N_f+1}$$

<u>example : $N_f = 3$</u>

Global symmetry is $SO(6)$

Dropping instanton part I^{inst}, one finds

$$I_{pert} = 1 + \left(e^{-im_1 - im_2} + \cdots + e^{im_2 + im_3} + 3 + 1\right)x^2 + O(x^3)$$

where the constant 1 is a singlet of $SO(6)$ and m_i are arranged to form the 15-dim adjoint representation of $SO(6)$:

$$e^{-im_1 - im_2} + \cdots + e^{im_2 + im_3} + 3$$

In terms of characters we have:

$$I = 1 + \left(1 + \chi_{15}^{SO(6)}\right) x^2 + \cdots$$

Taking into account the instantons gives

$$I = 1 + \Big(1 + \chi_{15}^{SO(6)} + \underbrace{q\,\chi_4^{SO(6)} + q^{-1}\chi_{\bar{4}}^{SO(6)}}_{\text{spinor reps.}}\Big) x^2 + \cdots$$

where the power of q represents instanton number.

This mirrors the embedding of $so(6)$ into $E_4 = su(5)$:

$$SU(5) \supset SO(6) \times U(1)_I$$

$$24 = 1_0 + 15_0 + 4_1 + \bar{4}_{-1}$$

In character notation we have:

$$I = 1 + \left(\chi_1^{SO(6)} + \chi_{15}^{SO(6)} + q\,\chi_4^{SO(6)} + q^{-1}\chi_{\bar{4}}^{SO(6)}\right) x^2 + \cdots$$

$$\equiv 1 + \chi_{24}^{E_4} x^2 + \cdots$$

This generalizes as follows:

$$I_{N_f} = 1 + \chi_{adj.}^{E_{N_f+1}} x^2 + \cdots$$

with the following embedding structure:

$$E_{N_f+1} \supset SO(2N_f) \times U(1)_I$$

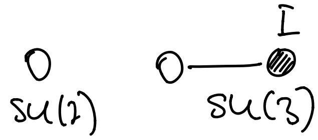
$$adj^{E_{N_f+1}} = 1_0^{SO(2N_f)} + adj_0^{SO(2N_f)} + 2_1^{N_f-1} + 2_{-1}^{\prime\, N_f-1}$$

We have:

$N_f = 2$: $E_3 = SU(3) \times SU(2) \supset SO(4) \times U(1)_I$

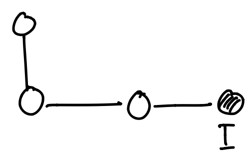
$SU(3) \supset SU(2) \times U(1)_I$

$8 = 1_0 + 3_0 + 2_1 + 2_{-1}$



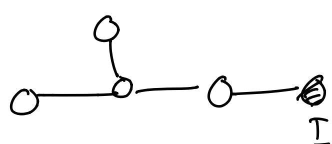
$N_f = 3$: $E_4 = SU(5) \supset SU(4) \times U(1)_I$

$24 = 1_0 + 15_0 + 4_1 + \bar{4}_{-1}$



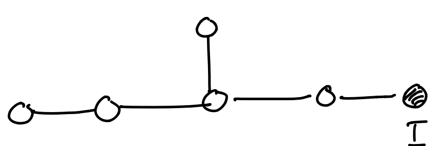
$N_f = 4$: $E_5 = SO(10) \supset SO(8) \times U(1)$

$45 = 1_0 + 28_0 + 8_{-1} + 8_1$



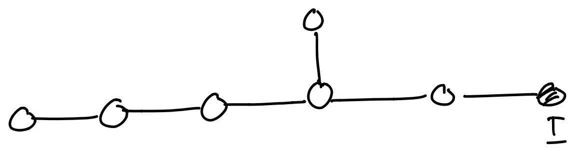
$N_f = 5$: $E_6 \supset SO(10) \times U(1)$

$78 = 1_0 + 45_0 + 16_{-1} + \overline{16}_1$



$N_f = 6$: $E_7 \supset SO(12) \times U(1)$

$133 = 1_0 + 66_0 + 32_1 + 32_{-1} + 1_2 + 1_{-2}$



$N_f = 7: \quad E_8 \supset SO(14) \times U(1)$

$$248 = 1_0 + 91_0 + 64_1 + \overline{64}_{-1} + 14_2 + 14_{-2}$$